Detailed Derivations

Mean Ergodicity

A w.s.s. process $\{u[n]\}$ is mean ergodic in the mean square error sense if $\lim_{N\to\infty} \mathbb{E}\left[|m - \hat{m}(N)|^2\right] = 0$

Question: under what condition will this be satisfied?

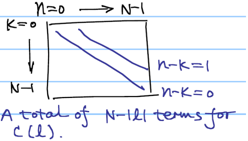
$$E\left[|\hat{m}(N) - m|^2\right] = E\left[\left|\frac{1}{N}\sum_{n=0}^{N-1} u[n] - m\right|^2\right]$$

$$= \frac{1}{N^2} E\left[\left|\sum_{n=0}^{N-1}(u[n] - m)\right|^2\right]$$

$$= \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} E\left[(u[n]-E(u[n]))(u[k]-E(u[k]))^*\right]$$

$$= \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} c(n-k)$$

$$\overset{l \triangleq n-k}{=} \frac{1}{N}\sum_{l=-N+1}^{N-1}\left(1 - \frac{|l|}{N}\right) c(l)$$



Therefore, the necessary and sufficient condition for $\{u[n]\}$ to be mean ergodic in MSE sense is

$$\lim_{N\to\infty} \frac{1}{N}\sum_{l=-N+1}^{N-1}\left(1 - \frac{|l|}{N}\right) c(l) = 0 \qquad [**]$$

Properties of $\mathbf{R}$

$\mathbf{R}$ is Hermitian, i.e., $\mathbf{R}^H = \mathbf{R}$

<u>Proof</u> $r(k) \triangleq \mathbb{E}\left[u[n]u^*[n-k]\right] = \left(E[u[n-k]u^*[n]]\right)^* = [r(-k)]^*$

Bring into the above $\mathbf{R}$, we have $\mathbf{R}^H = \mathbf{R}$

$\mathbf{R}$ is <u>Toeplitz</u>.

A matrix is said to be Toeplitz if all elements in the main diagonal are identical, and the elements in any other diagonal parallel to the main diagonal are identical.

$\mathbf{R}$ Toeplitz $\Leftrightarrow$ the w.s.s. property.

Properties of **R**

R is non-negative definite , i.e., $\underline{x}^H \mathbf{R} \underline{x} \geq 0$, $\forall \underline{x}$

Proof

Recall $\mathbf{R} \triangleq \mathbb{E}\left[\underline{u}[n]\underline{u}^H[n]\right]$. Now given $\forall \underline{x}$ (deterministic):

$$\underline{x}^H \mathbf{R} \underline{x} = \mathbb{E}\left[\underline{x}^H \underline{u}[n]\underline{u}^H[n]\underline{x}\right] = \mathbb{E}\left[\underbrace{(\underline{x}^H \underline{u}[n])}_{|x| \text{ scalar}}(\underline{x}^H \underline{u}[n])^*\right] =$$

$$\mathbb{E}\left[|\underline{x}^H \underline{u}[n]|^2\right] \geq 0$$

- eigenvalues of a Hermitian matrix are real.
  (similar relation in FT analysis: real in one domain becomes conjugate symmetric in another)
- eigenvalues of a non-negative definite matrix are non-negative.
  Proof choose $\underline{x} =$ **R**'s eigenvector $\underline{v}$ s.t. $\mathbf{R}\underline{v} = \lambda \underline{v}$,
  $\underline{v}^H \mathbf{R} \underline{v} = \underline{v}^H \lambda \underline{v} = \lambda \underline{v}^H \underline{v} = \lambda |v|^2 \geq 0 \Rightarrow \lambda \geq 0$

Properties of **R**

Recursive relations: correlation matrix for $(M+1)\times 1$ $\underline{u}[n]$:

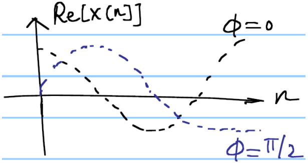
$$R_{M+1} = \begin{bmatrix} r(0) & r(1) & \cdots & r(M) \\ r^*(1) & r(0) & \cdots & r(M-1) \\ r^*(2) & r^*(1) & \ddots & \vdots \\ \vdots & \vdots & & \\ r^*(M) & r^*(M-1) & \cdots & r(0) \end{bmatrix} \quad \text{and} \quad \underline{u}_{M+1}[n] = \begin{bmatrix} \underline{u}_M[n] \\ \hline u(n-M) \end{bmatrix} = \begin{bmatrix} u[n] \\ \hline \underline{u}_M[n-1] \end{bmatrix}$$

$$= \left[\begin{array}{c|c} r(0) & \underline{r}^H \\ \hline \underline{r} & R_M \end{array}\right] = \left[\begin{array}{c|c} R_M & (\underline{r}^B)^* \\ \hline (\underline{r}^B)^T & r(0) \end{array}\right]$$

where $\underline{r} = \begin{bmatrix} r^*(1) \\ \vdots \\ r^*(M) \end{bmatrix}$, $\underline{r}^B = \begin{bmatrix} r^*(M) \\ \vdots \\ r^*(1) \end{bmatrix}$

(4) Example: Complex Sinusoidal Signal

$x[n] = A \exp[j(2\pi f_0 n + \phi)]$ where A and f_0 are real constant, $\phi \sim$ uniform distribution over $[0, 2\pi)$ (i.e., random phase)



We have:
$\mathbb{E}[x[n]] = 0 \ \forall n$

$\mathbb{E}[x[n]x^*[n-k]]$
$= \mathbb{E}[A \exp[j(2\pi f_0 n + \phi)] \cdot A \exp[-j(2\pi f_0 n - 2\pi f_0 k + \phi)]]$
$= A^2 \cdot \exp[j(2\pi f_0 k)]$

$\therefore$ $x[n]$ is zero-mean w.s.s. with $r_x(k) = A^2 \exp(j2\pi f_0 k)$.

Example: Complex Sinusoidal Signal with Noise

Let $y[n] = x[n] + w[n]$ where $w[n]$ is white Gaussian noise uncorrelated to $x[n]$, $w[n] \sim N(0, \sigma^2)$

Note: for <u>white noise</u>, $\mathbb{E}\left[w[n]w^*[n-k]\right] = \begin{cases} \sigma^2 & k = 0 \\ 0 & o.w. \end{cases}$

$$
\begin{aligned}
& r_y(k) = \mathbb{E}\left[y[n]y^*[n-k]\right] \\
& = \mathbb{E}\left[(x[n] + w[n])(x^*[n-k] + w^*[n-k])\right] \\
& = r_x[k] + r_w[k] \quad (\because \mathbb{E}\left[x[\cdot]w[\cdot]\right] = 0 \text{ uncorrelated and } w[\cdot] \text{ zero mean}) \\
& = A^2 \exp[j2\pi f_0 k] + \sigma^2 \delta[k]
\end{aligned}
$$

$$
\therefore \mathbf{R}_y = \mathbf{R}_x + \mathbf{R}_w = A^2 \underline{e}\underline{e}^H + \sigma^2 \mathbb{I}, \text{ where } \underline{e} = \begin{bmatrix} 1 \\ e^{-j2\pi f_0} \\ e^{-j4\pi f_0} \\ \vdots \\ e^{-j2\pi f_0 (M-1)} \end{bmatrix}
$$

Rank of Correlation Matrix

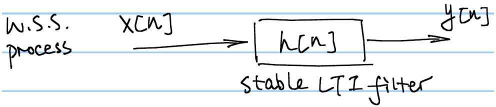
Questions:

The rank of $\mathbf{R}_x = 1$ ($\because$ only one independent row/column, corresponding to only one frequency component f_0 in the signal)

The rank of $\mathbf{R}_w = M$

The rank of $\mathbf{R}_y = M$

Filtering a Random Process



① $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[n-k]\,h[k]$

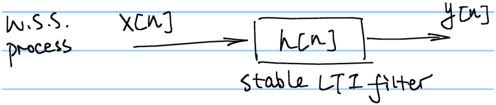
$E[y[n]] = m_x \sum_{k=-\infty}^{+\infty} h[k] = m_x H(\omega)\Big|_{\omega=0}$

② $r_{yx}(n+k, n) \triangleq E[\,y[n+k]\,x^*[n]\,] = E\Big[\sum_{l=-\infty}^{+\infty} x[n+k-l]\,h[l]\,x^*[n]\Big]$

$= \sum_{l=-\infty}^{+\infty} r_x(k-l)\,h[l]$ i.e. $r_{yx}(n+k, n)$ depends only on k and not on n.

$\Rightarrow \quad r_{yx}(k) = r_x(k) * h[k]$

Filtering a Random Process



⑶ $$r_y(n+k, n) = E[y[n+k]\, y^*[n]] = E\Big[y[n+k] \sum_{l=-\infty}^{+\infty} x^*[n-l]\, h^*[l]\Big]$$

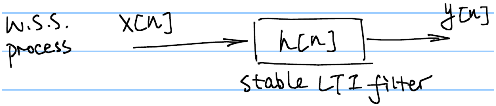
$$= \sum_{l=-\infty}^{+\infty} r_{yx}(k+l)\, h^*[l] \underset{l' \triangleq -l}{=} \sum_{l'=-\infty}^{+\infty} r_{yx}(k-l')\, h^*[-l']$$

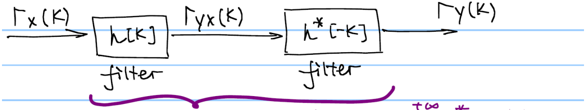
i.e. $E(y[n])$ & $r_y(\cdot)$ is not a func. of n $\Rightarrow$ $\{y[n]\}$ is w.s.s.

$$\Rightarrow \quad r_y(k) = r_{yx}(k) * h^*[-k] = r_x(k) * h[k] * h^*[-k]$$

$$= \sum_{l=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} h[l]\, h^*[-m]\, r_x(k-l-m)$$

Filtering a Random Process





$$\Gamma_h[k] = h[k] * h^*[-k] = \sum_{l=-\infty}^{+\infty} h^*[l]\, h[k+l]$$

deterministic autocorrelation of filter's impulse response